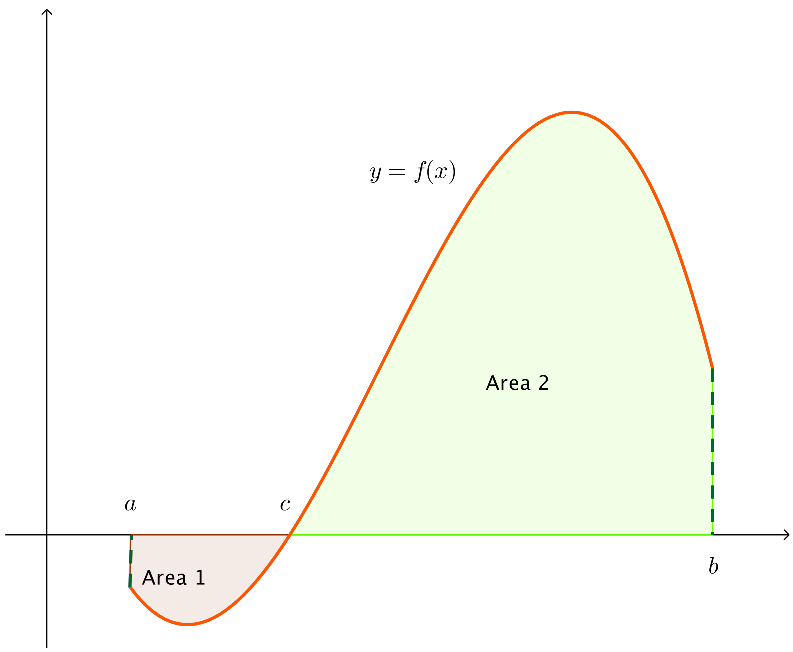
4 Applications of Definite Integration

4.1 Area bounded by the graph of $f(x)$ and the x-axis on $[a,b] = \int_a^b |f(x)|\,dx$



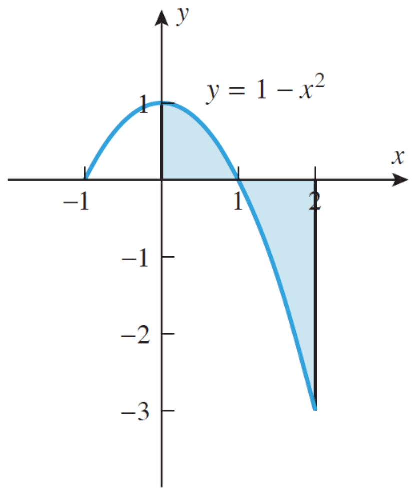
$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = -\text{ Area 1} + \text{ Area 2 } = \text{Signed area}}$$

$$\boxed{\int_a^b |f(x)|\,dx = \int_a^c -f(x)\,dx + \int_c^b f(x)\,dx = \text{ Area 1} + \text{ Area 2 } = \text{Area}}$$

Example 4.1. Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval $[0,2]$.

Solution. Let $1 - x^2 = 0, \quad \Rightarrow \quad x = \pm 1.$

$$1 - x^2 \begin{cases} \geq 0, & \text{for } -1 \leq x \leq 1, \\ < 0, & \text{for } x < -1 \quad \text{or} \quad x > 1. \end{cases}$$



$$|1-x^2| = \begin{cases} 1-x^2 & \text{when } 1-x^2 \geq 0 \\ x^2-1 & \text{otherwise} \end{cases}$$

$$1-x^2 \geq 0 \iff 1 \geq x^2$$

$$1 \geq x \geq -1$$

The area is given by

$$\begin{aligned}
\int_0^2 |1-x^2|\,dx &= \int_0^1 (1-x^2)dx + \int_1^2 -(1-x^2)dx \\
&= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2 = \left(1-\frac{1}{3}\right) - 0 - \left(2-\frac{8}{3}\right) + \left(1-\frac{1}{3}\right) \\
&= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2.
\end{aligned}$$

■

Exercise 4.1. Area bounded by the graph of $f(x) = x - \sqrt{x}$ and x-axis on $[0, 2]$.

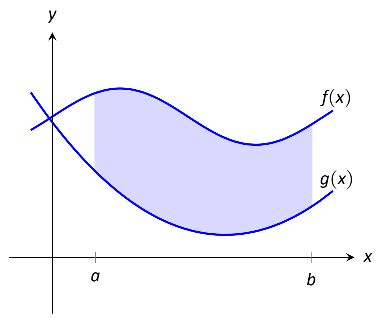
4.2 Area bounded by the graphs of $f(x)$, $g(x)$ on $[a, b]$ = $\int_a^b |f(x) - g(x)|\,dx$

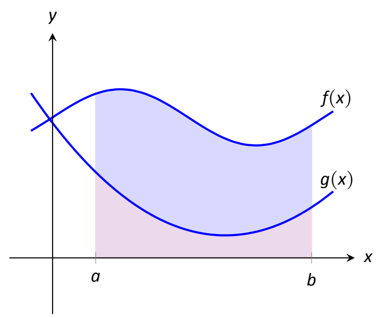
Theorem 4.1. *Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Proof. The area between $f(x)$ and $g(x)$ is obtained by subtracting the area under g from the area under f. Thus the area is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

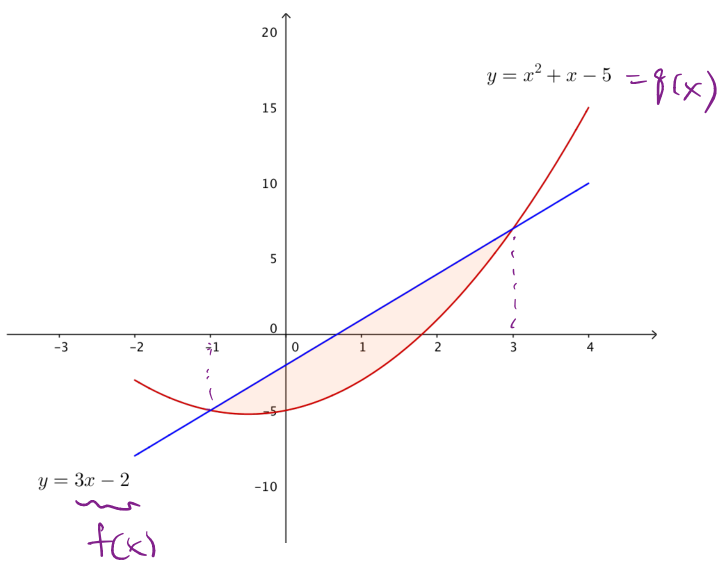




□

More generally, without the assumption $f \geq g$, area $= \int_a^b |f-g|\,dx$

Example 4.2. Find the area of the region enclosed by the curves $\underbrace{y = x^2+x-5}_{g(x)}$ and $\underbrace{y = 3x-2}_{f(x)}$ in the $x-y$ plane.



Solution. Let $x^2 + x - 5 = 3x - 2 \quad \Rightarrow \quad x = -1, 3.$

intersection points of $y = g(x)$ and $y = f(x)$: at an intersection point (x, y)

$y = x^2 + x - 5 = 3x - 2 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = 3, -1$

The area is

$$\begin{aligned}
\int_{-1}^{3} \big(\underbrace{(3x-2)}_{f} - \underbrace{(x^2+x-5)}_{g.}\big)\,dx &= \int_{-1}^{3} (-x^2+2x+3)\,dx \\
&= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Bigg|_{-1}^{3} \\
&= -\frac{1}{3}(27) + 9 + 9 - \left(\frac{1}{3} + 1 - 3\right) \\
&= 10\frac{2}{3}.
\end{aligned}$$

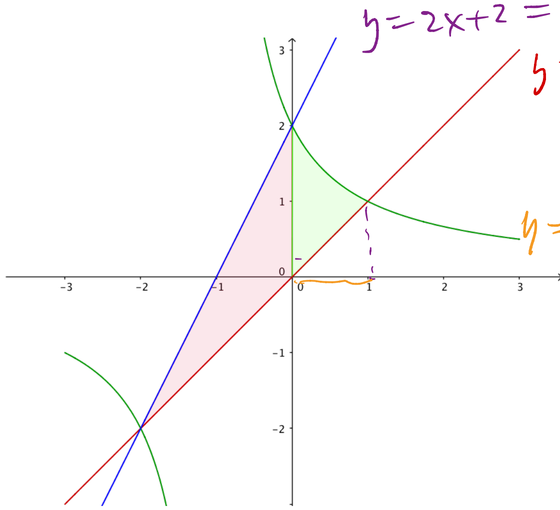
■

Example 4.3. Find the area bounded by the curves

$$y = f(x) = x, \quad y = g(x) = \frac{2}{x+1}, \quad \text{and} \quad y = h(x) = 2x + 2.$$

$y = 2x+2 = h(x)$

$y = x = f(x)$



$y = \frac{2}{x+1} = g(x)$

intersection of the blue and red lines:

$y = 2x+2 = x \Rightarrow x = -2$

intersection of blue and green curves:

$y = 2x+2 = \frac{2}{x+1}$

$2(x+1)^2 = 2 \Rightarrow x+1 = \pm 1$

$\Rightarrow x = 0, -2$

intersection of green and red curves

$y = \frac{2}{x+1} = x \Rightarrow x^2 + x = 2$

$\Rightarrow x = -2, 1$

Solution. Area is

$$\begin{aligned}
&\int_{-2}^{0} (h(x) - f(x))dx + \int_{0}^{1} (g(x) - f(x))dx \\
&= \int_{-2}^{0} (2x + 2 - x) + \int_{0}^{1} \left(\frac{2}{x+1} - x\right) dx \\
&= \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + \left[2\ln|x+1| - \frac{x^2}{2}\right]_{0}^{1} \\
&= 2 + (2\ln 2 - \frac{1}{2}) = \frac{3}{2} + \ln 4.
\end{aligned}$$

■

4.3 Other Applications

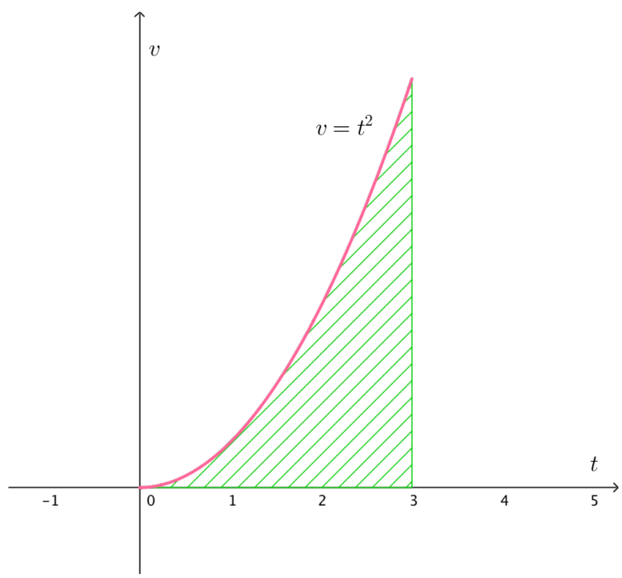
Example 4.4. An object moves along x-axis towards right with speed $v(t) = t^2$ m/s. Calculate the distance traveled from $t = 0$ to $t = 3$s.

Solution. Let $S(t)$ be the position at t. Then, $S'(t) = v(t) = t^2$.

The distance from $t = 0$ to $t = 3$ is

$$\underbrace{S(3) - S(0)}_{\text{total distance change}} = \int_0^3 \overbrace{S'(t)}^{\text{rate of change}} dt = \int_0^3 t^2\, dt = \frac{1}{3}t^3 \Big|_0^3 = 9\text{m}$$

Geometrically,



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Example 4.5. Let $L(t)$ be the level of carbon monoxide (CO). Given that $L'(t) = 0.1t + 0.1$ parts per million (ppm). How much will the pollution change from $t = 0$ to $t = 3$?

Solution.

$$L(3) - L(0) = \int_0^3 \underbrace{L'(t)}_{0.1t+0.1} dt = \left[0.05t^2 + 0.1t\right]_0^1 = 0.75\text{ppm}.$$

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Exercise 4.2. Let t be the time (in hour). Let $m(t)$ be the mass of a certain amount of protein. The protein is changed to an amino acid and cause a decrease in mass at a rate

$$\frac{dm}{dt} = \frac{-2}{t+1}\text{g/hr}.$$

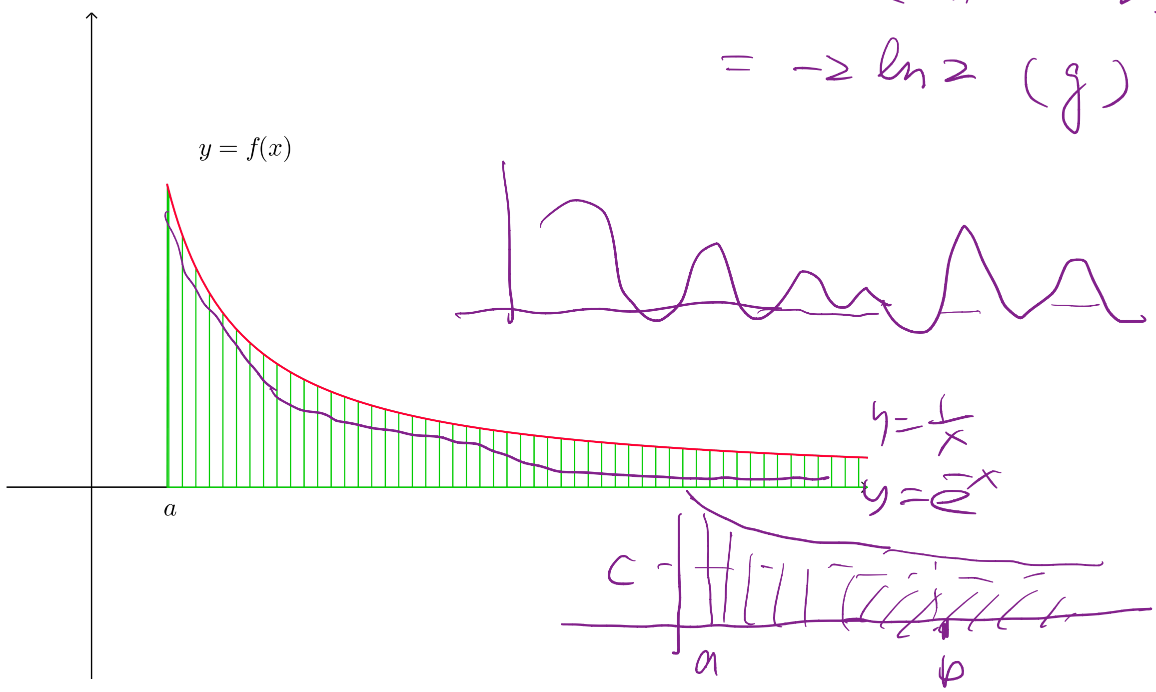
Find the decrease in mass of the protein from $t = 2$ to $t = 5$.

Ans: $-2\ln 2$.

$$m(5) - m(2) = \int_2^5 \frac{dm}{dt}\,dt = \int_2^5 \frac{-2}{t+1}\,dt$$
$$= -2 \ln|t+1|\Big|_2^5$$
$$= -2(\ln 6 - \ln 3)$$
$$= -2\ln 2 \ (g)$$

5 Improper Integrals

Question: How to find area of an unbounded region?



Definition 5.1. The following types of integrals are called "improper integrals" (of the first type). The integrals we have encountered previously, namely integrals of piecewise continuous functions over finite intervals, are "proper integrals".

Define

1.
$$\boxed{\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

Note: for this to converge $\Rightarrow \lim_{x\to+\infty} f(x) = 0$

but the converse is not true e.g. $\int_1^\infty \frac{1}{x}dx$

2.

$$\boxed{\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

3. Let c be a fixed real number.

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx}$$

if both the two integrals on the right are convergent, we say that the integral is **convergent**. Otherwise, **divergent**.

Example 5.1.

$-(e^{-b} - e^{0}) = 1 - e^{-b}$

1. $\int_0^{+\infty} e^{-x}dx = \lim_{b\to+\infty}\int_0^b e^{-x}dx = \lim_{b\to+\infty} -e^{-x}\big|_0^b = \lim_{b\to+\infty}(e^0 - e^{-b}) = \lim_{b\to+\infty}(1-e^{-b}) = 1$, convergent.

2. $\int_1^{+\infty}\frac{1}{x}dx = \lim_{b\to+\infty}\int_1^b\frac{1}{x}dx = \lim_{b\to+\infty}\ln x|_1^b = \lim_{b\to+\infty}(\ln b - \ln 1) = \lim_{b\to+\infty}\ln b = +\infty$, divergent.

$= -\left(\frac{1}{b} - \frac{1}{1}\right)$

3. $\int_1^{+\infty}\frac{1}{x^2}dx = \lim_{b\to+\infty}\int_1^b\frac{1}{x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to+\infty}\left(1-\frac{1}{b}\right) = 1$, convergent.

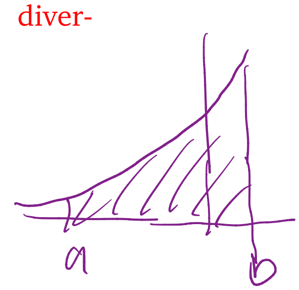
4. $\int_1^{+\infty}\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty}\int_1^b\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty}2\sqrt{x}\Big|_1^b = \lim_{b\to+\infty}2(\sqrt{b}-1) = +\infty$, divergent.

5. $\int_{-\infty}^{0} e^x dx = \lim_{a\to-\infty}\int_a^0 e^x dx = \lim_{a\to-\infty} e^x\Big|_a^0$

$= \lim_{a\to-\infty}(e^0 - e^a)$

$= 1 - 0 = 1$ □

convergent



Example 5.2. Compute $\displaystyle\int_0^{+\infty} \frac{dx}{(x+1)(3x+2)}$.

the integrand is a proper rational function. Partial fractions decomposition:

$$\frac{1}{(x+1)(3x+2)} = \frac{a}{x+1} + \frac{b}{3x+2}$$

$$= \frac{a(3x+2)}{(x+1)(3x+2)} + \frac{b(x+1)}{(3x+2)(x+1)}$$

$1 = a(3x+2) + b(x+1)$

when $x=-1$: $1 = a(-1) \Rightarrow a=-1$

when $x=-\frac{2}{3}$: $1 = b(\frac{1}{3}) \Rightarrow b=3$

Solution.

$$\frac{1}{(x+1)(3x+2)} = \frac{3}{3x+2} - \frac{1}{x+1}.$$

$u=3x+2$

Hence

$$\int_0^b \frac{dx}{(x+1)(3x+2)} = [\ln|3x+2| - \ln|x+1|]_0^b$$

$$= \ln|3b+2| - \ln|b+1| - \ln|2| = \ln\frac{|3b+2|}{|b+1|} - \ln 2.$$

Because

$$\lim_{b\to+\infty} \frac{|3b+2|}{|b+1|} = \lim_{b\to+\infty} \frac{|3b+2| \times \frac{1}{|b|}}{|b+1| \times \frac{1}{|b|}}.$$

$$\lim_{b\to+\infty} \frac{|3+\frac{2}{b}|}{|1+\frac{1}{b}|} = \frac{3}{1} = 3.$$

Therefore

$$\lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

$\lim_{x\to+\infty} \frac{1}{x^p} = 0$ when $p>0$, $\frac{1}{x^p}$ decays faster than $\frac{1}{x^q}$ as $x\to+\infty$ when $p>q$ in the sense $\lim_{x\to+\infty} \frac{\frac{1}{x^p}}{\frac{1}{x^q}} = 0$

Exercise 5.1. Let $p > 1$. Prove that

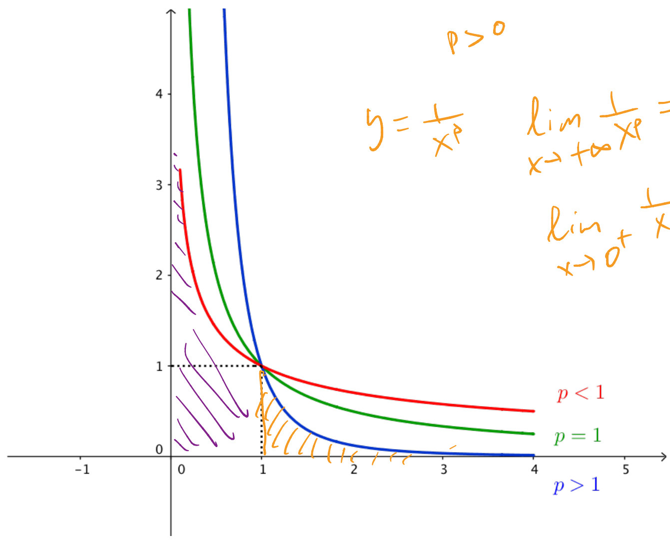
$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if} \quad p>1, \quad \text{convergent} \\ +\infty, & \text{if} \quad 0<p\le 1, \quad \text{divergent}. \end{cases}$$

$$= \lim_{b\to+\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to+\infty} \begin{cases} \dfrac{x^{1-p}}{1-p} & \text{when } p\neq 1 \\ \ln|x| & \text{when } p=1 \end{cases} = \begin{cases} \lim_{b\to+\infty} \left(\dfrac{1-\frac{1}{b^{p-1}}}{p-1} \right) \\ +\infty \end{cases}$$

Remark. From the above exercise,

1. $\lim_{x\to+\infty} f(x) = 0 \quad \not\Rightarrow \quad \displaystyle\int_1^{+\infty} f(x)\,dx$ is convergent.
2. For all $p>0$, $\frac{1}{x^p} \to 0$ as $x\to+\infty$. However, only for $p>1$, $\frac{1}{x^p}$ decays fast enough to guarantee the total area $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx$ is finite.

Remark. All the integration techniques can be applied, e.g. integration by substitution,...



$p > 0$

$y = \frac{1}{x^p}$ $\lim_{x \to +\infty} \frac{1}{x^p} = 0$

$\lim_{x \to 0^+} \frac{1}{x^p} = \infty$

Example 5.3. Compute $\int_{-\infty}^{1} xe^x \, dx$. (integration by parts)

Solution.

$$\int_{-\infty}^{1} xe^x \, dx = \lim_{a \to -\infty} \int_a^1 xe^x \, dx.$$

$u = x$ $\frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x$ $v = e^x$

$$\int \underbrace{x}_{u} \underbrace{e^x}_{\frac{dv}{dx}} \, dx = \int x d(e^x) = xe^x - \int e^x \, dx = (x-1)e^x + C.$$

$$\int_{-\infty}^{1} xe^x \, dx = \lim_{a \to -\infty} (x-1)e^x|_a^1 = (1-1)e^1 - (a-1)e^a$$

$$= \lim_{a \to -\infty} (1-a)e^a \quad \infty \cdot 0 \quad \text{indeterminate form}$$

$$= \lim_{a \to -\infty} \frac{1-a}{e^{-a}} \quad \frac{\infty}{\infty}$$

$$= \lim_{a \to -\infty} \frac{-1}{-e^{-a}} \quad \text{L'Hôpital's rule}$$

$$= 0.$$

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Exercise 5.2. $\int_{-\infty}^{1} x^2 e^x dx = e$ integration by parts twice

apply L'Hôpital's rule twice

Example 5.4. Compute $\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx$. (integration by substitution)

Solution. Using the substitution $u = 1 + x^2$, we have

$u = 1 + x^2$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$, $u(0) = 1$, $u(b) = 1 + b^2$

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{-1}{2(1+x^2)} + C.$$

Thus

$$\int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}$$

$\lim_{b\to+\infty} \int_0^b \frac{x\,dx}{(1+x^2)^2} = \lim_{b\to+\infty} \int_1^{1+b^2} \frac{\frac{1}{2}du}{u^2} = \lim -\frac{1}{2u}\Big|_1^{1+b^2} = \frac{1}{2}\left(\lim -\frac{1}{1+b^2} + 1\right)$

and

$$\int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}.$$

$\lim_{a\to-\infty} \int_a^0 \frac{x\,dx}{(1+x^2)^2} = \lim_{a\to-\infty}\left(-\frac{1}{2}\frac{1}{u}\right)\Big|_{1+a^2}^{1} = \lim_{a\to-\infty}\left(\frac{1}{2}\frac{1}{u}\right)\Big|_1^{1+a^2} = \frac{1}{2}\left(\lim_{a\to-\infty}\frac{1}{1+a^2} - 1\right)$

Hence

$$\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx + \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

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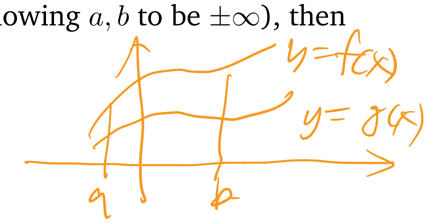
Fact: If $0 \le f(x) \le g(x)$ on the interval of integration (a, b) (allowing a, b to be $\pm\infty$), then



- If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.
- If $\int_a^b f(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

Example 5.5. Determine whether $\int_0^\infty x^n e^{-x}dx$ is convergent.

E.g. $\int_1^\infty \frac{e^{-x}}{x}dx$ is convergent because over $(1,\infty)$ $\frac{e^{-x}}{x} \le e^{-x}$ and $\int_1^\infty e^{-x}dx$ is convergent

$= \int_0^\infty e^{-x}dx - \int_0^1 e^{-x}dx$

Definition 5.2 (Improper integrals of Type 2)**.** The improper integrals defined in Definition 5.1 has infinite intervals of integration, but the values of the integrand are finite on the intervals of the integration. We also generalize definite integrals where the integrand may go to $\pm\infty$ over the interval of integration.

Suppose that $f(x)$ is continuous on (a, b), but $\lim_{x\to b^-} f(x) = \pm\infty$. Then we define:

$$\int_a^b f(x)dx := \lim_{c\to b^-} \int_a^c f(x)dx.$$

c < b
f is continuous
on [a, c]

Similarly, if $\lim_{x\to a^+} f(x) = \pm\infty$,

$$\int_a^b f(x)dx := \lim_{c\to a^+} \int_c^b f(x)dx.$$

Example 5.6. 1. $\displaystyle\int_0^1 \frac{1}{x^p}dx$

2. $\displaystyle\int_0^1 \frac{1}{\ln x}dx$

3. (mixed type) $\displaystyle\int_{-\infty}^1 \frac{1}{x^3}dx$